Addendum
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Abstract

This document describes further results related to my dissertation.

1 Higher-order logic as a query language

Recall that the main open conjecture of chapter 3 is the semantics preservation of the translation \( \| \) : \( \text{HOL} \rightarrow \text{NRC} \) for every domain-independent HOL sequent \( \Gamma \vdash e : t \)

\[ \| \Gamma \vdash e : t \| = \| [\Gamma \vdash e : t] \| \]

This conjecture is still open, but there are two new results to report:

1. This conjecture is false for the internal language of a topos (defined in section 3.2.2).
2. This conjecture is false for HOL + weakening.

1.1 Topoi

The following theorems are proved in Coq. Let \( L \) denote the internal language of a topos, as defined in section 3.2.2. Our translation \( \| \) : \( \text{HOL} \rightarrow \text{NRA} \) extends directly to \( \| \) : \( L \rightarrow \text{NRA} \).

Theorem (Topos-Eta).

\[ \| [\text{id}] \| = \| [\lambda \text{ev}] \| \]

Theorem (Topos-Beta-Weak1). Let \( A \times B : f : \Omega \) be a term in the internal language of a topos. Then

\[ \| [(\text{id}, y); f] \| = \| [\lambda (f, y); \text{ev}] \| \]

Theorem (Topos-Beta-Weak2). Let \( A \times B : f : \Omega \) be a term in the internal language of a topos. Then

\[ \forall I \in [A], J \in [B], (I, J) \in [f] \implies \text{atoms}(J) \subseteq \text{atoms}(I) \]

implies

\[ \| [(\pi_1; \lambda f, \pi_2); \text{ev}] \| = \| [f] \| \]

The condition above guarantees that \( \| \lambda f \| \) will contain no constants not in \( [f] \). If we violate this condition, the theorem is not true:

Theorem (Topos-No-Beta). There exists a hereditarily domain-independent \( f \), such as \( 1 \times D : \top : \Omega \), st

\[ \| [(\pi_1; \lambda f, \pi_2); \text{ev}] \| \neq \| [f] \| \]

Using the above, and that \( \| (\pi_1; \lambda f, \pi_2); \text{ev} \| = [f] \), and that \( \| \) is semantics preserving for hereditarily domain-independent terms, we obtain that

Theorem (Topos-No-Sem). There exists a domain independent \( f \), such as \( (\pi_1; \lambda \top, \pi_2); \text{ev} \), such that

\[ [f] \neq [f] \]
1.2 HOL + weakening

We can re-cast the development of the previous section in terms of HOL, if we add an additional typing rule:

\[
\begin{align*}
\text{WEAKEN} & \quad \Gamma \vdash e : t \quad x \text{ fresh} \\
\Gamma, x : s \vdash e : t & \\
\Gamma, x : s \vdash e : t \quad \text{WEAKEN-TRANS} & \quad \Gamma \vdash e : [t] \\
\Gamma, x : s \vdash e : t & \\
\Gamma \vdash e : [t] & \quad \text{WEAKEN-SEM} \\
\Gamma \vdash e : [t] & \quad \[\Gamma, x : s \vdash e : t\] = \pi_1; [\Gamma \vdash e : t]
\end{align*}
\]

Whereas in HOL, every sequent corresponds to a unique typing derivation, and vice-versa, with the addition of weakening, HOL sequents can have more than one derivation. Since the meaning of a sequent is a function of its typing derivation, HOL sequents can now potentially have more than one meaning; i.e., \[\] must be defined as a relation. Typically, we would prove a coherence theorem for HOL+weakening, stating that \[\] is a functional relation; i.e., regardless of which derivation we use to compute the meaning of a sequent, the meaning will be the same. We would need to use the coherence theorem to prove, for example, that the translation \[\] : \(HOL + \text{weakening} \rightarrow NRC\) is semantics preserving for hereditarily domain-independent derivations. However, for the purposes of showing that \[\] : \(HOL + \text{weakening} \rightarrow NRC\) is not semantics preserving for domain-independent terms, all we need to do is exhibit a counter-example derivation.

Consider:

\[
\begin{align*}
\frac{y : D \vdash \top : 2}{\frac{\lambda y : D. \top : D \rightarrow 2}{\text{ABS}}} & \\
\frac{x : D \vdash \lambda y : D. \top : D \rightarrow 2}{\frac{x \vdash D \vdash \lambda y : D. D \rightarrow D}{\text{WEAKEN}}} & \\
\frac{x \vdash D \vdash \lambda y : D. \top : D \rightarrow D}{\frac{\lambda y : D. \top}{\text{beta}}} & \frac{x \vdash D \vdash \lambda y : D. \top : D \rightarrow D}{\frac{x \vdash D \vdash x : D}{\text{VAR1}}} & \frac{x \vdash D \vdash x : D}{\frac{x \vdash D \vdash \lambda y : D. \top : D \rightarrow D}{\text{APP}}}
\end{align*}
\]

Setting \([D] = \{c\}\), we find that (ignoring superfluous units):

\[
\begin{align*}
\frac{[[y : D \vdash \top : 2]](c \mapsto \top)}{\frac{[[\lambda y : D. \top : D \rightarrow 2]](\langle \rangle \mapsto \{\})}{\text{ABS}}} & \\
\frac{[[x : D \vdash \lambda y : D. \top : D \rightarrow 2]](c \mapsto \{\})}{\frac{[[x : D \vdash \lambda y : D. \top : D \rightarrow 2]](c \mapsto \{\})}{\text{WEAKEN}}} & \\
\frac{[[x : D \vdash D \vdash \lambda y : D. \top : D \rightarrow D]](c \mapsto c)}{\frac{[[x : D \vdash x : D]](c \mapsto c)}{\text{VAR1}}} & \\
\frac{[[x : D \vdash x : D]](c \mapsto \bot)}{\text{APP}}
\end{align*}
\]

We conclude that \([[y : D \vdash \top : 2]](c \mapsto \top)\) and \([[x : D \vdash (\lambda y : D. \top) \vdash x : D \vdash D \rightarrow 2]](c \mapsto \bot)\). As before, if we assume \([\] is semantics preserving for domain-independent terms, we have a contradiction by noting that \([[y : D \vdash \top : 2]] = [[x : D \vdash (\lambda y : D. \top) : x : D \vdash D \rightarrow 2]]\) (beta) and \([[\top]] = [[\top]]\).

It is also the case that \([\] : \(HOL + \text{weakening} \rightarrow NRC\) is incoherent: using a typing derivation that does not including weakening, we have that \([[x : D \vdash (\lambda y : D. \top) : x : D \vdash D \rightarrow 2]](c \mapsto \top)\). This is why this counter-example does not apply to pure HOL.

Finally, it is worth noting that, if you consider \([\] as a translation from HOL to the internal language of a topos (L, as defined in 3.2.2), then there is no HOL sequent that maps to \((\pi_1; \Lambda f, \pi_2); ev\). To see this, note that the only way to obtain \(\pi_1; \Lambda f\) is by rule VAR1, which implies that \(\Lambda f\) must be a sequent of projections, which is impossible. Weakening allows us to obtain \(\pi_1; \Lambda f\) by a rule other than VAR1 (namely, WEAKEN).

We conjecture that \([\] : \(HOL + \text{weakening} \rightarrow NRC\) is both coherent and semantics preserving for hereditarily domain-independent terms.